Moore-Penrose’s Inverse and Solutions of Linear Systems

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Abstract: We employ the generalized inverse matrix of Moore-Penrose to study the existence and uniqueness of the solutions for over- and under-determined linear systems, in harmony with the least squares method.

Key words: Linear systems - SVD - Least squares technique - Pseudoinverse of Moore-Penrose

INTRODUCTION

For any real matrix $A_{nxm}$, Lanczos \cite{1, 2} introduces the matrix:

$$S_{(n+m)\times(n+m)} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

with $A^T$ denoting the transpose matrix and studies the eigenvalue problem:

$$S \vec{\omega} = \lambda \vec{\omega},$$

where the proper values are real because $S$ is a real symmetric matrix. Besides:

$$\text{rank } A = p = \text{Number of positive eigenvalues of } S,$$

such that $1 \leq p \leq \min (n, m)$ Then the singular values or canonical multipliers follow the scheme:

$$\lambda_1, \lambda_2, \ldots, \lambda_p, -\lambda_1, -\lambda_2, \ldots, -\lambda_p, 0, 0, \ldots, 0,$$

that is, $\lambda = 0$ has the multiplicity $n + m - 2p$. Only in the case $p = n = m$ can occur the absence of the null eigenvalue.

The proper vectors of $S$, named ‘essential axes’ by Lanczos, can be written in the form:

$$\vec{\omega}(n+m)_{x1} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}^T,$$

then (1) and (2) imply the Modified Eigenvalue Problem:

$$A_{nxm} \vec{v}_{mx1} = \lambda \vec{u}_{nx1}, \quad A^T_{nxm} \vec{u}_{nx1} = \lambda \vec{v}_{mx1},$$

hence:

$$A^T A \vec{v} = \lambda^2 \vec{v}, \quad A A^T \vec{u} = \lambda^2 \vec{u},$$

with special interest in the associated vectors with the positive eigenvalues because they permit to introduce the matrices:

$$U_{nxp} = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p), \quad V_{mxp} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p),$$

verifying $U^U = V^V = b_{np}$ because:

$$\vec{u}_j \cdot \vec{u}_k = \vec{v}_j \cdot \vec{v}_k = \delta_{jk},$$

therefore $\vec{\omega}_j \cdot \vec{\omega}_k = 2 \delta_{jk}, j, k = 1, 2, \ldots, p$. Thus, the Singular Value Decomposition (SVD) express \cite{1-5} that $A$ is the product of three matrices:

$$A_{nxm} = U_{nxp} A_{pp} V_{mxp}^T = \text{Diag} (\lambda_1, \lambda_2, \ldots, \lambda_p).$$

This relation tells that in the construction of $A$ we do not need information about the null proper value; the information from $\lambda = 0$ is important to study the existence and uniqueness of the solutions for a linear system associated to $A$. Golub \cite{6} mentions that the SVD has played a very important role in computations, in solving least squares problems \cite{7}, in signal processing problems and so on; it is just a very simple decomposition, yet it is of fundamental importance in many problems arising in technology.
It is important to observe that the symmetric matrices $(UU^T)_{nn}$ and $(VV^T)_{mm}$ are identity matrices for arbitrary vectors into their respective spaces of activation [5], that is:

$$UU^T u = u, \quad \forall u \in Col U, \quad VV^T v = v, \quad \forall v \in Col V; \quad (11)$$

besides, (10) allows obtain the SVD of the Gram matrices:

$$(AA^T)_{mn} = U \Lambda^2 U^T, \quad (A^T A)_{mm} = V \Lambda^2 V^T, \quad (12)$$

such that $p = rank A = rank (AA^T) = rank (A^T A)$.

From (10) and (12) we observe that:

$$Col A = Col (AA^T) = Col U, \quad Col A^T = Col (A^T A) = Col V. \quad (13)$$

The eigenvectors associated with $\lambda = 0$ verify the equations:

$$\tilde{v}_j = 0, \quad j = 1,\ldots,m-p, \quad A^T \tilde{u}_k = 0, \quad k = 1,\ldots,n-p, \quad (14)$$

$$\tilde{v}_r \tilde{v}_j = 0, \quad \forall r,j, \quad \tilde{u}_t \tilde{u}_k = 0, \quad \forall t,k$$

therefore:

$$V^T v_j = 0, \quad \forall j, \quad U^T u_k = 0, \quad \forall k, \quad (15)$$

$$A^T \tilde{x} \in Col U \quad \text{and} \quad A^T \tilde{x} \in Col V, \quad \forall \tilde{x} \in E^m, \quad A^T \tilde{y} \in Col V \quad \text{and} \quad A^T \tilde{y} \in Col U, \quad \forall \tilde{y} \in E^n. \quad (16)$$

In Sec. 2 we exhibit the Moore-Penrose’s pseudoinverse of $A$ [8-13] via the corresponding SVD [14-16], which is useful in Sec. 3 to study the solutions of over- and under-determined linear systems [2, 5] in the spirit of the least squares method [7, 17].

**Generalized Inverse:** The Moore-Penrose’s inverse [2, 8-13] is given by:

$$A^+ = (UU^T)^{-1} U^T, \quad (17)$$

which coincides with the natural inverse obtained by Lanczos [2, 5]. The matrix (16) satisfies the relations [10, 11, 13]:

$$(AA^+)^T = AA^+ = A^+, \quad (AA^+)^T = AA^+, \quad (A^+ A)^T = A^+ A, \quad (18)$$

that characterize the pseudoinverse of Moore-Penrose. In particular, from (10), (11) and (16):

$$A A^+ = U U^T \quad : \quad A A^+ u = u, \quad \forall u \in Col U, \quad (19)$$

$$A^+ A = V V^T \quad : \quad A^+ A v = v, \quad \forall v \in Col V. \quad (20)$$

The use of (8) and (10) into (16) implies the following expression for the Lanczos generalized inverse:

$$A^+ = (\tilde{u}_1\tilde{u}_2\ldots\tilde{u}_n), \quad t_j = u^{(j)}_1 v^T + u^{(j)}_2 v_2 + \ldots + u^{(j)}_p v_p, \quad j = 1,\ldots,n, \quad (21)$$

where $u^{(j)}_k$ means the $j$-th component of $\tilde{u}_k$; similarly:

$$(A^+)^T = (r_1r_2\ldots r_p), \quad r_k = \frac{v^{(k)}_1}{\lambda_1} u_1 + \frac{v^{(k)}_2}{\lambda_2} u_2 + \ldots + \frac{v^{(k)}_p}{\lambda_p} u_p, \quad k = 1,\ldots,m, \quad (22)$$

therefore:

$$Col A^+ = Col V, \quad Col (A^+)^T = Col (U \Lambda^{-1} V^T) = Col U. \quad (23)$$

We can use (16) to construct the pseudoinverse of each Gram matrix, in fact [13]:

$$(A^T A)_{m x m} = V \Lambda^{-2} V^T, \quad (AA^T)_{m x m} = U \Lambda^{-2} U^T, \quad (24)$$

with the interesting properties:

$$(A^T A)^+ A = A^+, \quad (A A^+)^+ A = (A^+)^T, \quad (A^T A)^+ (A^T A) = A^+ A = V V^T. \quad (25)$$

Each matrix has a unique inverse because every matrix is complete within its own spaces of activation. The activated $p$-dimensional subspaces (eigenspaces / operational spaces) are uniquely associated with the given matrix [5].

**Linear Systems:** We want to find $\tilde{x} \in E^m$ verifying the linear system:

$$A \tilde{x} = \tilde{b}, \quad (26)$$

for the data $A_{nn}$ and $\tilde{b} \in E^n$. It is convenient to consider two situations:
a). Over-determined linear system [2, 5]: In this case we have more equations than unknowns, that is, \( m < n \).

Lanczos [18] comments that the ingenious method of least squares makes it possible to adjust an arbitrarily over-determined and incompatible set of equations. The problem of minimizing \((Ax - \tilde{b})^2\) has always a definite solution, no matter how compatible or incompatible the given system is. The least square solution of (24) satisfies [5, 17]:

\[
A^T A \tilde{x} = A^T \tilde{b}, \quad \tilde{x} \in Col V, \quad p = m, \tag{25}
\]

and the remarkable fact about (25) is that it always gives an even-determined (balanced) system, no matter how strongly over-determined the original system has been.

The system (25) is compatible because from (13) and (15) we have that \( A^T \tilde{b} \) is into \( Col (A^T A) = Col V \). Now we multiply (25) by \( (A^T A)^+ \) and we use (11) and (23) to obtain the solution:

\[
\tilde{x} = A^+ \tilde{b}, \tag{26}
\]

which is unique because \( p = m \), that is, \( Col V = E^n \), then in (14) the system \( A^+ \tilde{b} \) only has the trivial solution; hence the Moore-Penrose’s inverse gives the least square solution of (24). The expression (26) is in harmony with the results in [19-22].

We have eliminated over-determination (and possibly incompatibility) by the method of multiplying both sides of (24) by \( A^T \). The unique solution thus obtained coincides with the solution generated with the help of \( A^T \) [5].

b). Under-determined linear system [2, 5]: There are more unknowns than equations, that is, \( n < m \).

In this case we may try the least square formulation of (24), that is, to accept (26), however, now the solution is not unique because \( p < m \) and the system \( A^+ \tilde{b} \) has \( m - p \) non-trivial independent solutions; an under-determined system remains thus under-determined, even in the least square approach.

An alternative process is to transform the original \( \tilde{x} \) into the new unknown \( \tilde{z} \) via the relation [5]:

\[
\tilde{x} = A^T \tilde{z}, \tag{27}
\]

then (24) acquires the structure \( AA^T \tilde{z} = \tilde{b} \) whose least square solution is given by the pseudoinverse of Moore-Penrose:

\[
\tilde{z} = (AA^T)^+ \tilde{b} + \sum_{j=1}^{n-p} c_j \tilde{z}_j, \tag{28}
\]

where the quantities \( c_j \) are arbitrary and the \( \tilde{z}_j \) are \( n - p \) independent vectors generating the Kernel \((AA^T) = Kernel (A) [13] \), that is:

\[
A^T \tilde{z}_j = 0, \quad j = 1, \ldots, n - p. \tag{29}
\]

Thus, from (16), (22), (23), (28) and (29) we have that the solution of (27) is given by:

\[
\tilde{x} = A^T (AA^T)^+ \tilde{b} = V \Lambda^{-1} U^T \tilde{b} = A^+ \tilde{b},
\]

in agreement with (26).

Although that (26) is not unique for the under-determined case, we can say that it is the ‘natural solution’ for the linear system (24).

CONCLUSIONS

Our study shows the importance of the SVD [1-6, 14-16] of a matrix and of the corresponding Moore-Penrose’s inverse [8-13], to elucidate the least square solution [7, 17-22] for over- and under-determined linear systems [2, 5].

REFERENCES